HEDGING THE STANDARD OF LIVING VIA COST OF LIVING INDEX FUTURES

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Abstract

People dislike inflation because inflation erodes the real value of future nominal income and wealth. Adjustment of future nominal values via a cost of living index is an appropriate way to handle the problem of real income risk. Nonetheless an important aspect needs more discussion: If markets existed in which 'real income risks' could be traded—would a rational individual always voluntarily purchase protection against such risk? A model is developed to shed some light on this aspect. It shows that the optimal behaviour depends—as expected—on the cost of protection and the risk preferences of the individual.

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1 Introduction

There is an ongoing discussion in many OECD countries about the future of public pension programs. The existing pension arrangements are far too costly in many countries and future payments of programs seem to be uncertain. There are several proposals under discussion to reform the current public pension arrangements. A currently widely debated proposal in Germany picks up the foreseeable financial restrictions of the public pension plan in the future and wants to encourage greater "Eigenvorsorge" (i.e., private pension schemes). Even though

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there are differences between the government and the opposition about the details of how to encourage people to save more for their retirement, the goal as such is undisputed. People should accumulate funds until the end of their working live. Upon retirement accumulated funds would be converted into annuity payments. Thus, these private annuities partially take over the role of social security pensions.

However, such annuities and social security payments are not perfect substitutes. While the latter are indexed with a *Cost of Living Index* (COLI)—and thus try to guarantee real payments—most annuities are nominal. Therefore, private annuities possess real income risk. Using public survey methods Robert Shiller (1997) has found that people's main concern about inflation is the erosion of the real value of future income. Consequently it is conceivable that people might be interested in financial instruments that would allow them to swap nominal payments into real payments. Or, more directly, that they are interested in *real* annuities. For example, in the Chilean social security system retirees could choose between nominal pension payments and payments that are adjusted via the *Unidad de Fomento* (Schulz-Weidner 1999, Shiller 1998). The Unidad de Formento is in effect a COLI. Thus, we should enlarge the debate about a reform of public pension programs on the discussion of financial instruments that allow retirees to choose between real and nominal pension payments.

Nevertheless, there remains a large obstacle to discuss such instruments. This obstacle is the failure of the CPI futures market that was originated in 1986 at the Coffee, Sugar and Cocoa Exchange in New York (Horrigan 1987). Economists were convinced that such a contract "would in a variety of ways help reduce the hardship created by uncertain future purchasing power" (Lovell and Vogel 1973, p.1010). And Milton Friedman conjectured in 1986 that CPI futures could become the "largest-volume contract in the country" (cited from Horrigan 1987, p.11). But they and many others were wrong: the market failed due to a lack of interest and the CPI futures market closed in 1991 (Wrase 1997). *Are there fundamental theoretical reasons for the failure of the COLI futures market?*

I use a simple model to address this question. The purpose of my model is to take a closer look at the behaviour of risk averse rational individuals under real income risk who have the possibility of hedging their risk by means of a futures contract. There exists a close relationship between risk aversion in income and behaviour towards price risk (Hanoch 1977). This is remarkable because it shows that an individual's behaviour towards price risk is by no means clear-cut. Even if an individual loves price risk, it is not certain that he would try to increase the spread of his standard of living between states. Altogether, I derive the following results: transactions at a COLI futures market will only occur if risk averse individuals have different degrees of risk aversion. However, this implies that not all risk averse individuals reduce their real income risk.

Some of them increase their real income risk. Furthermore, I show that even those individuals who wish to cancel out any real income risk can do this only if they have homothetic preferences and if the traded index is exact for their preferences.

The paper is organized as follows: First, I describe the general setup of the model. After that section, I derive some conclusions for a simple model with two states and proportional price vectors. The conclusions are valid without any restrictions on the utility functions of the individuals. In the subsequent section the model is extended to allow for more than two states. In this framework a complete hedge is generally only possible, when preferences are homothetic and the traded index is proportional to the true COLI (Konüs index). An intuitive reason for this result suggests itself immediately: as is well known, the Konüs index is independent of any reference utility for homothetic preferences. Finally, I show for indirect utility functions with different degrees of constant relative risk aversion that the futures demand is decreasing in the futures price, that there exists a price to induce a complete hedge, and that there is a price at which the futures demand is equal to zero. In the concluding section I give alternative explanations for the failure of the COLI futures market.

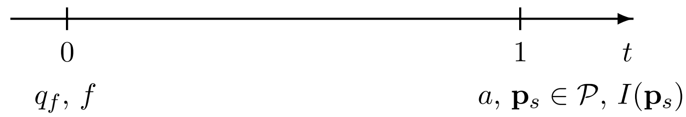
2 The model

2.1 Assumptions

We assume there are just two periods $t \in \{0, 1\}$. In period 0 a COLI index futures contract is traded which settles in period 1. The price per contract is f. In period 1 the individual receives a fixed nominal payment $a > 0$ (a fixed rent) and the settlement payments from his engagement at the futures market. With this income the individual can buy in period 1 a consumption bundle out of $L \geqslant 2$ different goods. However, the prices of these goods—comprised in the vector $\mathbf{p} \in \mathbb{R}^L_{++}$—are uncertain. There are $S \geqslant 2$ different price vectors which could occur in period 1. These price vectors are collected in $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_S\}$. The probability π_s for state s is positive and all S probabilities sum to 1. The COLI—see subsection 2.2 for some examples—will be calculated in period 1 with the realized price vector $\mathbf{p}_s \in \mathcal{P}$. The formula of the COLI is common knowledge. Let $I(\mathbf{p})$ denote the COLI and let q_f denote the number of futures contracts. Then the settlement payment in period 1 per contract is $(I(\mathbf{p}_s) - f)$ for a *long position* (q_f is positive and the individual *buys* contracts in period 0) and $-(I(\mathbf{p}_s) - f)$ for a *short position* (q_f is negative and the individual *sells* contracts in period 1). The total income of the individual in $t = 1$ in state s is

$$y_s = a + q_f\big(I(\mathbf{p}_s) - f\big) . \tag{1}$$

The chronological order of the model can be illustrated as follows:



To derive meaningful results, we have to restrict the admissible futures prices f. Let I_{min} denote the minimal index value and let I_{max} denote the maximal index value (for all $\mathbf{p} \in \mathcal{P}$). Then it must always be the case that

$$f \in (I_{min}, I_{max}) \ . \tag{2}$$

Otherwise it is possible to earn riskless profits (for example: if $f \leqslant I_{min}$ the payment for a long position is never negative).

Now we can look at the optimal behaviour of the individual in period 0. In this period the individual decides about his futures demand q_f given his expectations about the π_s's and his utility function. I assume that the expected direct utility is of the following additive form

$$U \equiv \sum_{s=1}^{S} \pi_s u(\mathbf{x}_s) \ . \tag{3}$$

Under this assumption (and some regularity conditions for the direct utility function) the maximization problem is easily transformed into the equivalent problem of maximization of the *indirect* utility function. To show this equivalence, one writes down the Lagrangian for (3) with the $2S$ constraints $y_s = \mathbf{p}_s \cdot \mathbf{x}_s$ and $y_s = a + q_f(I_s - f)$. With the help of the first order conditions it is easily seen that this maximization is in effect a "two stage process": In every state we divide given income optimally among the goods (first stage gives the state indirect utility) and then divide income optimally between states.

So let us assume from now on that the individual has a twice-differentiable indirect (state) utility function $v(\mathbf{p}, y)$. This function is homogeneous of degree zero in $\mathbf{p}$ and y together (no money illusion) and increasing in y. Furthermore we assume

$$v_{yy}(\mathbf{p}, y) \leqslant 0 \ . \tag{4}$$

If $v_{yy} = 0$, the individual is *risk neutral*, for which case the indirect utility function will be of the form $v(\mathbf{p}, y) \equiv \tilde{v}(\mathbf{p})y$. If $v_{yy} < 0$, the individual is *risk averse*. In that case we further assume that

$$\lim_{y \to 0} v_y(\mathbf{p}, y) \to \infty \ . \tag{5}$$

This assumption guarantees that the optimal choice of q_f always corresponds to $y_s \geqslant 0$ for all states. Now we can put *real income risk* in concrete terms: such a risk exists if

$$\exists\ \mathbf{p}_s, \mathbf{p}_{s'} \in \mathcal{P}:\ v(\mathbf{p}_s, a) \neq v(\mathbf{p}_{s'}, a)\ . \tag{6}$$

The real value of income a depends on the realization of the price vector. Real income risk is non-existent if

$$\forall\ \mathbf{p}_s \in \mathcal{P}:\ v(\mathbf{p}_s, a) = const. \tag{7}$$

Now we are ready to look on the behaviour of the individual. The individual chooses his futures demand via the maximization of

$$U \equiv E\big[v(\mathbf{p}, y)\big] = \sum_{s=1}^{S} \pi_s v(\mathbf{p}_s, y_s) \tag{8}$$

with the income definition (1).

Before we solve this problem explicitly, what should we expect? First of all we might expect that a risk averse individual tries to minimize his real income risk. But risk averse with respect to what? What if the individual loves price risk? As Hanoch (1977) has shown, an individual could be risk averse with respect to income (given prices) *and* risk loving (given income) with respect to prices. We can state this more concretely:

Lemma 1. *If the indirect utility $v(\mathbf{p}, y)$ is convex in $\mathbf{p}$, then for the* relative risk aversion *(RRA)*

$$RRA(\mathbf{p}, y) \equiv -\frac{v_{yy}(\mathbf{p}, y)y}{v_y(\mathbf{p}, y)}\ , \tag{9}$$

$RRA(\mathbf{p}, y) \leqslant 2$ is sufficient for the convexity of $v(\mathbf{p}, y)$ in $\mathbf{p}$, only if preferences are homothetic (Hanoch 1977:419 (Theorem 3)).

A price risk lover is an individual whose indirect utility function is convex in $\mathbf{p}$. He always prefers an uncertain distribution of price vectors as against the certain average of the prices. Does this mean that an individual with a convex indirect utility function is not interested in COLI futures? Intuition suggests the following answer: COLI futures make it possible to reallocate income between different states. The price vectors in these states remain unaffected by this reallocation. It is not possible to equalize the price vectors between states. So even price risk loving individuals might be interested in COLI futures.

2.2 The COLI

Which are the relevant *cost of living indexes*? The most prominent one is the Laspeyres price index

$$I^l \equiv \frac{\mathbf{p}^T \mathbf{x}_R}{\mathbf{p}_R^T \mathbf{x}_R} \ . \tag{10}$$

$\mathbf{x}_R$ is a bundle of goods (basket), which represents the consumption pattern of a reference individual (or household). The CPI and the *Deutsche Preisindex für die Lebenshaltung* are calculated in this fashion. It is a well-known result that such indexes overstate the difference between true costs of living associated with non-proportional vectors $\mathbf{p}_R$ and $\mathbf{p}$. This overstatement is due to the fact that substitution effects are neglected. Thus the Laspeyres price index is an upper bound for the *true* COLI

$$I^k \equiv \frac{e(\mathbf{p}, u_R)}{e(\mathbf{p}_R, u_R)} \ , \tag{11}$$

with $e(\mathbf{p}, u)$ as the expenditure function. The index (11)—which we term also as *Konüs* index—would be exact for the reference individual. There are many other index formulas. They all have in common that they are linear homogeneous in prices.

3 First results with simplifying assumptions

Let be $\mathcal{P} = \{\underline{\mathbf{p}}, \overline{\mathbf{p}}\}$ with

$$\overline{\mathbf{p}} \equiv \alpha \underline{\mathbf{p}} \quad \text{and} \quad \alpha > 1 \ . \tag{12}$$

Due to $\overline{\mathbf{p}} > \underline{\mathbf{p}}$ there is always—independently of the concrete utility function—a real income risk in the sense of condition (6). Let $\pi \equiv Prob\left(\underline{\mathbf{p}}\right) \in (0,1)$ denote the probability of the low price vector. The linear homogeneous COLI can take the two values $\underline{I} \equiv I(\underline{\mathbf{p}})$ and $\overline{I} \equiv I(\overline{\mathbf{p}})$ with

$$\overline{I} = \alpha \underline{I} > \underline{I} \ . \tag{13}$$

The income in the different states is $\underline{y} \equiv a + q_f(\underline{I} - f)$ and $\overline{y} \equiv a + q_f(\overline{I} - f)$. If the individual chooses $q_f = a/f$ his income is $\overline{y} = \alpha \underline{y}$ and so his standard of living is constant

$$v(\underline{\mathbf{p}}, \underline{y}) = v(\alpha \underline{\mathbf{p}}, \alpha \underline{y}) \ . \tag{14}$$

We see that it is always possible to hedge completely.

The maximization problem is given as

$$\max_{q_f} \ U \equiv \pi v(\underline{\mathbf{p}}, \underline{y}) + (1 - \pi) v(\overline{\mathbf{p}}, \overline{y}) \ . \tag{15}$$

The first order condition for a risk averse individual is

$$-\pi v_y(\underline{\mathbf{p}}, \underline{y})(\underline{I} - f) = (1-\pi) v_y(\overline{\mathbf{p}}, \overline{y})(\overline{I} - f) \ . \tag{16}$$

Let us concentrate on only two possible futures prices. In Appendix A.1 we give the reasoning behind this decision.

The first price we choose is $f = E[I]$. If we use this price we can transform the first order condition (16) into

$$v_y(\underline{\mathbf{p}}, \underline{y}) = v_y(\overline{\mathbf{p}}, \overline{y}) \ . \tag{17}$$

Depending on the behaviour of the marginal utility in prices, we can derive with $v_{yy} < 0$ that $\overline{y} > \underline{y}$ for increasing marginal utility, $\overline{y} = \underline{y}$ for constant marginal utility and finally $\overline{y} < \underline{y}$ for decreasing marginal utility. With this result at hand we can easily derive the corresponding optimal futures demand as $q_f^* > 0$, $q_f^* = 0$ and $q_f^* < 0$. In Appendix A.2 we show that

$$RRA(\underline{\mathbf{p}}, y) \begin{cases} > 1 & \forall \ y \in [a/\alpha, a] & \Rightarrow & q_f^* > 0 \\ = 1 & \forall \ y \in [a/\alpha, a] & \Rightarrow & q_f^* = 0 \\ < 1 & \forall \ y \in [a/\alpha, a] & \Rightarrow & q_f^* < 0 \end{cases} \ . \tag{18}$$

Thus we have seen that: with only two possible proportional price vectors the hedging behaviour depends directly on the degree of risk aversion in income. Furthermore, we can now answer the question that we posed at the end of the last section: Even if an individual loves price risk (his indirect utility function is convex in prices), it is possible for him to reduce his real income risk (see Lemma 1). The borderline case is RRA equal to one – which is equivalent to constancy of the marginal utility of income (see Appendix A.2). We see finally that $f = E[I]$ could also be an equilibrium price for the futures market even if *all* individuals are risk averse. For this case it is only necessary that the individuals have different degrees of risk aversion.

The second price of interest is given as

$$f^{ch} = \Big(E[1/I]\Big)^{-1} \ . \tag{19}$$

It is not difficult to check that $f^{ch} \in (I_{min}, I_{max})$ and $f^{ch} < E[I]$. Under the price (19) every risk averse individual chooses a *complete hedge*. This is easy to see: we obtain from the first order condition (16) $\alpha v_y(\underline{\mathbf{p}}, \underline{y}) = v_y(\overline{\mathbf{p}}, \overline{y})$. With $\overline{\mathbf{p}} = \alpha \underline{\mathbf{p}}$ and the homogeneity of degree minus one it follows that

$$v_y(\underline{\mathbf{p}}, \underline{y}) = v_y(\underline{\mathbf{p}}, \overline{y}/\alpha) \ , \tag{20}$$

and so $\overline{y} = \alpha \underline{y}$. Thus, the real income is equal in both states. This result is quite easy to understand when we assume that the two states are equally likely. Applying this assumption we obtain from (19)

$$-\alpha(\underline{I} - f^{ch}) = (\overline{I} - f^{ch}) \; . \tag{21}$$

Hence it is possible to swap real income one-for-one between states. However, at the price f^{ch} every risk averse individual chooses a complete hedge. Thus, whenever only risk averse individuals are operating in the futures market, this price will not be an equilibrium price.

Let us summarize our results under the simplifying assumptions: If the futures price is such that the real income can be exchanged one-for-one between states, a risk averse individual will hedge completely. In this case the standard of living is equalized across states. If the futures price is such that the nominal income can be exchanged one for one between states, the result will no longer be definite. The optimal futures demand then depends on the behaviour of relative risk aversion along a proportional price ray. It follows that a complete hedging of the standard of living will never be optimal and indeed there are cases when no hedging whatsoever would be chosen.

4 The general model with more than two states

4.1 Complete hedge of the standard of living

Now we allow for $S \geqslant 3$ possible states. A complete hedge means

$$v(\mathbf{p}_s, y_s) = u \qquad \forall \;\; \mathbf{p}_s \in \mathcal{P}. \tag{22}$$

We obtain

Proposition 1. *A complete hedge is only possible for arbitrary $\mathcal{P}$ and $S \geqslant 3$, $f > 0$ and $a > 0$ if*

$$q_f = \frac{a}{f} \; . \tag{23}$$

Furthermore, $I(\mathbf{p})$ has to be linear homogeneous, which means that for $\alpha > 0$

$$I(\alpha \mathbf{p}) = \alpha I(\mathbf{p}) \; . \tag{24}$$

Finally we must have that

$$I(\mathbf{p}) \propto e(\mathbf{p}, u) \; . \tag{25}$$

$e(\mathbf{p}, u)$ is the expenditure function that corresponds to $v(\mathbf{p}, y)$. All these are necessary conditions.

Proof. Condition (22) defines implicitly $e(\mathbf{p}, u)$. So

$$a + q_f \left(I(\mathbf{p}_s) - f \right) = e(\mathbf{p}_s, u) \qquad \forall\ \mathbf{p}_s \in \mathcal{P}\ . \tag{26}$$

For arbitrary $\mathcal{P}$ we must have that $q_f \neq 0$. We obtain with the help of (26)

$$(a - q_f f) + q_f I(\mathbf{p}_s) = e(\mathbf{p}_s, u) \qquad \forall\ \mathbf{p}_s \in \mathcal{P}\ . \tag{27}$$

We know that the RHS is linear homogeneous in prices. Because $\mathcal{P}$ is arbitrary, the first term on the LHS (a constant) must be equal to zero and the index must be linear homogeneous. If follows that

$$\frac{a}{f} I(\mathbf{p}_s) = e(\mathbf{p}_s, u) \qquad \forall\ \mathbf{p}_s \in \mathcal{P}\ . \tag{28}$$

and so the index has to be proportional to the expenditure function. ■

With (23) the total nominal income a is swapped into real income. It follows from the proportionality of the index with the expenditure function that the index is also proportional to the Konüs index. To make things simple, we assume from now on that the index is equal to the Konüs index. We obtain

Proposition 2. *Even if the traded index is the Konüs index (11) and the necessary conditions of Proposition 1 are fulfilled, a complete hedge for arbitrary a, f and u_R is only feasible, if and only if preferences are homothetic. Homothetic preferences are given by the following utility function*

$$H\big(\tilde{u}(\mathbf{x})\big)\ , \quad H'(\cdot) > 0\ , \tag{29}$$

where $\tilde{u}(\mathbf{x})$ is linear homogeneous.

Proof. With Proposition 1 the income is

$$y_s = \frac{a}{f} \frac{e(\mathbf{p}_s, u_R)}{e(\mathbf{p}_R, u_R)} \qquad \forall\ \mathbf{p}_s \in \mathcal{P}\ . \tag{30}$$

Define

$$\delta \equiv \frac{a}{f e(\mathbf{p}_R, u_R)}\ . \tag{31}$$

A complete hedge—irrespective of the distribution of the prices, δ and u_R—means that

$$v\big(\mathbf{p}_s, \delta e(\mathbf{p}_s, u_R)\big) = f(u_R, \delta) \qquad \forall\, \mathbf{p}_s \in \mathcal{P}\ . \tag{32}$$

For the special case $\delta = 1$ this condition is always fulfilled. Because $\mathcal{P}$ is arbitrary, (32) must be true for every feasible $\mathbf{p}$.

Sufficient condition: Homothetic preferences fulfill (32). To show this, we use the expenditure function and derive the indirect utility function. For this, we transform the expenditure function for homothetic preferences to get

$$\begin{aligned} e(\mathbf{p}, u) &\equiv \min_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \mid H(\tilde{u}(\mathbf{x})) \geq u\} \\ &= \min_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \mid \tilde{u}(\mathbf{x}) \geq H^{-1}(u)\} \\ &= \min_{\mathbf{x}} \left\{ H^{-1}(u)\mathbf{p}' \frac{\mathbf{x}}{H^{-1}(u)} \,\middle|\, \tilde{u}\left(\frac{\mathbf{x}}{H^{-1}(u)}\right) \geq 1 \right\} \\ &= H^{-1}(u) \min_{\hat{\mathbf{x}}} \{\mathbf{p}'\hat{\mathbf{x}} \mid \tilde{u}(\hat{\mathbf{x}}) \geq 1\} \\ &= H^{-1}(u)c(\mathbf{p}) \; . \end{aligned} \tag{33}$$

$c(\mathbf{p})$ is the *unit cost function*, which is increasing, concave, and linear homogeneous. Invert the expenditure function to get the indirect utility function

$$v(\mathbf{p}, y) = H\left(\frac{y}{c(\mathbf{p})}\right) . \tag{34}$$

We receive with (33) and (34)

$$v\big(\mathbf{p}, \delta e(\mathbf{p}, u_R)\big) \equiv H\left(\delta H^{-1}(u_R)\right) \; . \tag{35}$$

and so (32).

Necessary condition: If (32) holds for arbitrary $\mathbf{p}$, we derive after differentiation with respect to p_l

$$-\frac{v_{p_l}\big(\mathbf{p}, \delta e(\mathbf{p}, u_R)\big)}{v_y\big(\mathbf{p}, \delta e(\mathbf{p}, u_R)\big)} = \delta e_{p_l}(\mathbf{p}, u_R) \qquad l = 1...L \; . \tag{36}$$

We obtain with the help of Roy's identity, Shepard's Lemma and $h_l(\mathbf{p}, u) = x_l\big(\mathbf{p}, e(\mathbf{p}, u)\big)$

$$x_l\big(\mathbf{p}, \delta e(\mathbf{p}, u_R)\big) = \delta x_l\big(\mathbf{p}, e(\mathbf{p}, u_R)\big) \qquad l = 1...L \; . \tag{37}$$

Every utility function that fulfills (32) has demand functions that are linear homogeneous in income. However, this is equivalent to the statement that preferences are homothetic (Lau 1970, Theorem XI). This result completes our proof. ■

Corollar 1. *The possibility to hedge completely does not depend on u_R. The Konüs index for homothetic preferences is independent of u_R.*

Proof. We obtain immediately for (35) with (31) and (33)

$$v\big(\mathbf{p}, \delta e(\mathbf{p}, u_R)\big) = H\left(\frac{a}{fc(\mathbf{p}_R)}\right) . \tag{38}$$

We obtain also with (33)

$$I^k = \frac{c(\mathbf{p})}{c(\mathbf{p}_R)} . \tag{39}$$

■

This result is not difficult to interpret: to hedge completely means that the real income is in every possible state identical. To transform the nominal income y_s (30) into real income, we have to discount it with the COLI. For homothetic preferences the COLI is a function only of prices. y_s is proportional to the COLI. Thus, it follows that

$$y_s^{real} = \frac{a}{f} \qquad \forall\, \mathbf{p}_s \in \mathcal{P} . \tag{40}$$

Now we can come back to the question that we asked at the end of the introductory section: If a COLI is traded, will it be used? It seems quite difficult to answer this question in general terms. We consider only a particular specification of the utility function that has a prominent place in the literature.

5 The general case with constant relative risk aversion

We assume that the utility function is of the following form

$$v(\mathbf{p}, y) \equiv \frac{1}{1-r}\left(\frac{y}{c(\mathbf{p})}\right)^{1-r} \tag{41a}$$

for $r > 0$, $r \neq 1$ and

$$v(\mathbf{p}, y) \equiv \ln y - \ln c(\mathbf{p}) \tag{41b}$$

for $r = 1$. These functions are generated by homothetic preferences. If $r \leqslant 2$ the utility function is convex in $\mathbf{p}$ (see Lemma 1 and Appendix A.3). The Konüs index is equal to $c(\mathbf{p})/c(\mathbf{p}_R)$. The futures are based on this index. So it is possible—if so desired—to hedge completely.

We can simplify notation by defining $c_s \equiv c(\mathbf{p}_s)$ for all $\mathbf{p}_s \in \mathcal{P}$. All possible S realizations are comprised in $\mathcal{C}$. We designate the minimal expenditures in $\mathcal{C}$ as c_{min} and the maximal expenditures as c_{max}. We write the Konüs index with $c_R \equiv c(\mathbf{p}_R)$ shortly as $I_s^k \equiv c_s/c_R$. We have $I_{min}^k \equiv c_{min}/c_R$ and $I_{max}^k \equiv c_{max}/c_R$. Optimal futures demand $q_f^*(f, r, a)$ is derived from the problem of maximizing the expected utility (8) given $\mathcal{C}$. The solution to this problem must satisfy the first order condition

$$E\left[c^{r-1}y^{-r}(I^k - f)\right] = 0\;. \tag{42}$$

It is easy to check that the marginal utility of income goes towards infinity if $y_s \to 0$. Thus, income in every possible state is therefore non-negative. Let us state some results about the demand function, which we prove in Appendix A.4:

- Optimal futures demand is linear homogeneous in a, so that

  $$q_f^*(f,r,a) = q_f^*(f,r)a \tag{43}$$

  with $q_f^*(f,r) \equiv q_f^*(f,r,1)$.

- There exists a price

  $$f^{ch} \equiv \left(E\left[1/I^k\right]\right)^{-1} < E\left[I^k\right] \quad \text{and} \quad I^k_{min} < f^{ch} \tag{44}$$

  which induces a complete hedge. This price is independent of r.

- Futures demand is decreasing in f, which means that

  $$\frac{\partial q_f^*(f,r,a)}{\partial f} < 0\;. \tag{45}$$

- For every r there exists a price $f^0(r)$ for that $q_f^* = 0$. We have

  $$f^0(r) = E\left[\frac{c^{r-1}}{E\left[c^{r-1}\right]}\,I^k\right] \quad \text{with} \quad f^0(r) \gtreqless E\left[I^k\right] \;\;\Leftrightarrow\;\; r \gtreqless 1\;. \tag{46}$$

  Furthermore we have $f^0(r_h) > f^0(r_l)$ for $r_h > r_l$ and $I_{min} < f^0(r) < I_{max}$.

- We can conclude with the last two conditions that

  $$q_f^*\left(E[I^k], r\right) \gtreqless 0 \;\;\Leftrightarrow\;\; r \gtreqless 1\;. \tag{47}$$

Figure 1 shows three simplified demand functions $q_f^*(f,r)$ for individuals with different degrees of risk aversion. To make things simple it is supposed that every individual receives a fixed pension $a = 1$. At f^{ch} every individual hedges completely.

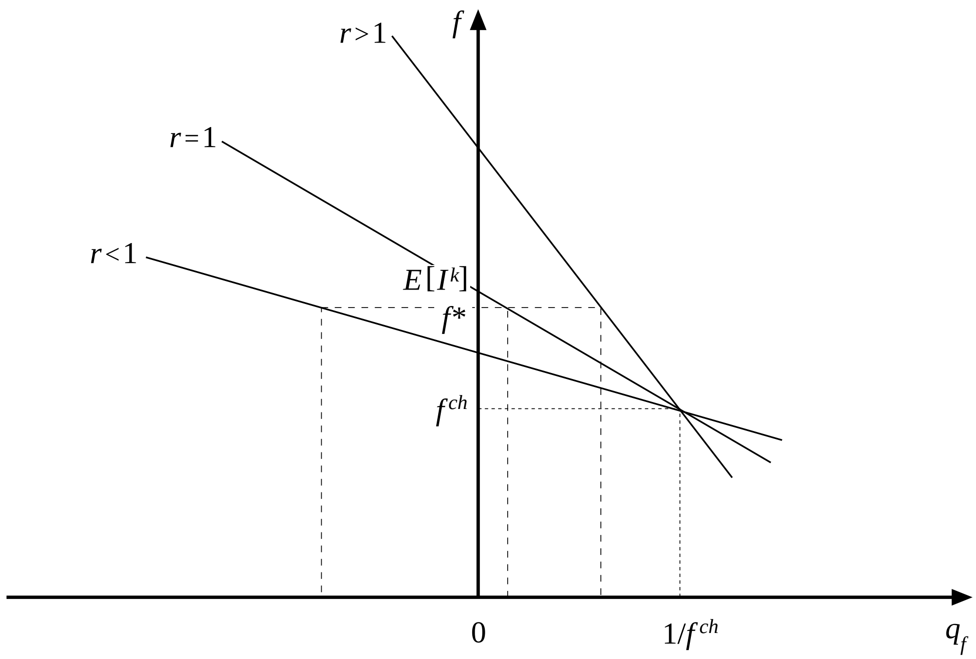


Figure 1

Futures demand with different degrees of risk aversion

If all individuals are risk averse this price can not be an equilibrium price. At $f = E[I^k]$ the demand function of an individual with $r = 1$ cuts the price axis. In that case $q^* = 0$. An individual with $r > 1$ buys contracts and an individual with $r < 1$ sells contracts. Finally, the market clearing price is equal to f^*: The individual with $r < 1$ supplies at that price a number of contracts which is equal to the demand of the other two individuals.

It is easy to extend the analysis to $I \geqslant 2$ individuals with different fixed pensions a_i and with different degrees of relative risk aversion r_i. The excess demand function at the futures market is given as

$$z(f) = \sum_{i=1}^{I} q_f^*(f, r_i) a_i \ . \tag{48}$$

The market clearing price is given implicitly by $z(f^*) = 0$. Due to $z'(f) < 0$ the market is stable. If $r_i = r$ for all individuals, the only market clearing price is $f^0(r)$ and no transactions happen. Transactions only happen if the individuals have different degrees of relative risk aversion. Finally, the price f^{ch} can never be market clearing.

6 Conclusion

Our model shows that there are no fundamental theoretical reasons why a COLI futures market could not succeed. A market equilibrium is possible even if all individuals are risk averse. However, one question remains: why did the CPI futures market fail? There exist different explanations for this fact:

- **Existence of substitutional strategies**: There were other possible strategies that could have been used as a hedge against real income risk. Such an alternative is easy to illustrate in the framework of our model: If we assume that capital markets are complete and short selling is not restricted, then individuals could borrow in $t = 0$ the amount $a/(1+i)$ (with i as the certain interest rate) and could buy a portfolio of assets, which generates in every possible state an equal amount of real income. To see this, let π_s denote the Arrow-Debreu prices and H_s the number of assets that pay in state s one unit of money. Thus, we have

$$\frac{a}{1+i} = \sum_{s}^{S} H_s \pi_s \quad \text{and} \quad H_s = y^{real} I_s^k \quad \text{for} \quad s = 1...S \ .$$

  $y^{real} > 0$ is the real income in state s. These are $S+1$ variables and equations, so that the system is solvable. However, this argument is not really a strong one because in reality one seldom (if ever) observes people borrowing against their future pension payments.

- **Failure of retailing**: The volume of the CPI futures contract was too large for the average—potential—user. Furthermore, the time horizon of maximal 3 years was too short.

- **Psychological barriers**: The introduction of the CPI futures in 1985 was ill-timed because most people were inexperienced with indexed financial instruments. Today, such instruments are perfectly common.

Our main results are: heterogeneous rational individuals are interested to participate in a COLI futures market. Nevertheless, this does not mean that every individual will reduce his real income risk with such futures. But even an individual who increases his real income risk does this to increase his welfare. Furthermore, we have seen that it will be nearly impossible to construct COLI's which allow for complete hedges under every conceivable constellation. Indeed we derived these results only under simplifying assumptions. Nevertheless, if we allow for more than two periods, the results are easily transferable with the assumptions of additive utility and separate futures contracts for the different time horizons.

In the introductory part of this paper we mentioned the currently ongoing debate about proposals to reform public pension systems. An issue in this debate ought to be financial instruments that permit individuals to hedge real income risk. We have not yet found theoretical reasons against such instruments. They would rather extend the individual possibilities to provide for retirement. However, our model is only a partial equilibrium model. Thus, it is necessary to extend the framework in the direction of a general equilibrium model in the next time.

A Appendix

A.1 Motivation for the prices we chose

$f = E[I]$. We can motivate this price as an equilibrium price with the following argument: we suppose that there are—in addition to risk averse individuals—risk neutral individuals active in the futures market. Furthermore, we assume that the prices which are an argument of the indirect utility function of the risk neutral individuals are independently distributed of the prices which are used to calculate the price index. We designate with $\mathbf{p}'$ the prices that influence the indirect utility of a risk neutral individual. $\mathcal{P}'$ is the set of all possible price realizations. The expected utility is equal to

$$U \equiv E_{\mathcal{P}'}[\tilde{v}(\mathbf{p}')] \left(\pi \underline{y} + (1-\pi)\overline{y}\right) \ . \tag{49}$$

For $f < E[I]$ a risk neutral individual puts his whole income into a long position and for $f > E[I]$ into a short position. So $E[I] = f$ is a plausible equilibrium price if there are many of such risk neutral individuals or if these individuals have huge income reserves.

$f = 1/E[1/I]$. We can motivate this price as an equilibrium price with the following argument: Suppose that risk neutral individuals are active at the futures market. Their real income depends on the prices in $\mathcal{P}$. With the help of the proportionality assumption we can write the expected utility as

$$U \equiv \tilde{v}(\underline{\mathbf{p}}) \left(\pi \underline{y} + (1-\pi)\overline{y}/\alpha\right) \ . \tag{50}$$

Such an individual puts all of his income into a long position if $f > f^{ch}$. For $f < f^{ch}$ he puts all his income into a short position. Thus f^{ch} is a plausible equilibrium price, if there are many of such risk neutral individuals or if these individuals have huge incomes.

A.2 Proof of (18)

How does the marginal utility of income change, if prices vary proportionally? The degree of homogeneity of v is zero. So v_y is homogeneous of degree minus one and we obtain with $\tilde{\alpha} > 0$

$$v_y(\tilde{\alpha}\mathbf{p}, y) = \frac{1}{\tilde{\alpha}} v_y(\mathbf{p}, y/\tilde{\alpha}) \tag{51}$$

and so

$$\frac{dv_y(\tilde{\alpha}\mathbf{p}, y)}{d\tilde{\alpha}} \gtreqless 0 \quad \Leftrightarrow \quad RRA(\mathbf{p}, y/\tilde{\alpha}) \gtreqless 1 \ . \tag{52}$$

Now let us use this result: If the degree of relative risk aversion (given $\underline{\mathbf{p}}$ and a) is greater one for all $\tilde{\alpha} \in [1, \alpha]$, then the marginal utility is a strictly increasing function on this support. It follows

$$v_y(\underline{\mathbf{p}}, a) < v_y(\alpha\underline{\mathbf{p}}, a) \tag{53}$$

and due to the decreasing marginal utility of income we obtain $q_f^* > 0$. The argument is analogous for the other cases.

Constancy of the marginal utility of income is equivalent with a constant degree of relative risk aversion equal to one. The explanation is as follows: the form of a cardinal utility function with constant marginal utility is $u(\mathbf{x}) \equiv \beta_0 + \beta_1 \ln \tilde{u}(\mathbf{x})$ (with $\beta_1 > 0$, $\tilde{u}$ is linear homogeneous and positive (Samuelson 1942:84)). This is a homothetic function and the indirect form is (see (34))

$$v(\mathbf{p}, y) = \beta_0 + \beta_1(\ln y - \ln c(\mathbf{p})) \; . \tag{54}$$

If we integrate $RRA(\mathbf{p}, y) = 1$, we obtain exactly this function (Hanoch 1977, p. 424) (remembering that Roy's identity is positive).

A.3 Convexity of the CRRA utility function

We have to prove that the CRRA utility function is convex in $\mathbf{p}$ for $r \leqslant 2$. Thus, in normalized prices we must have with $\mathbf{p}' \neq \mathbf{p}''$ and $\pi \in (0, 1)$

$$\frac{1}{1-r}\left(\pi c(\mathbf{p}')^{r-1} + (1-\pi)c(\mathbf{p}'')^{r-1}\right) \geqslant \frac{1}{1-r}\, c(\pi\mathbf{p}' + (1-\pi)\mathbf{p}'')^{r-1} \tag{55a}$$

for $r \neq 1$ and

$$\pi \ln c(\mathbf{p}') + (1-\pi)\ln c(\mathbf{p}'') \leqslant \ln c(\pi\mathbf{p}' + (1-\pi)\mathbf{p}'') \tag{55b}$$

for $r = 1$. We know that $c(\mathbf{p})$ is positive and concave. For $r = 1$ this function is transformed with the monotone increasing concave function $\ln c$. Thus it follows that $\ln c(\mathbf{p})$ is concave in $\mathbf{p}$ (Berck and Sydsæter 1993, 12.11). So the second inequality (55b) is true. For $1 < r \leqslant 2$ we have $1 - r < 0$. We make use of this fact and derive from (55a) a concavity condition for the function $H(c) \equiv c^{r-1}$. This function is increasing and concave for $1 < r \leqslant 2$, so that we can use the above mentioned argument. When we multiply the whole condition with -1, we derive a concavity condition for the function $\tilde{H}(c) \equiv -c^{r-1}$. This function is once again monotone increasing and concave for $r < 1$. Thus we can use the argument too for all cases with $r < 1$.

A.4 Characteristics of the futures demand function

The linear homogeneity is shown as follows: write the first order condition (42) as

$$E\left[c^{r-1}(1 + g(f, r, a)(I^k - f))^{-r}(I^k - f)\right] = 0 \tag{56}$$

with $g(f, r, a) \equiv q_f^*(f, r, a)/a$. It follows from the expression (56) that $\partial g/\partial a = 0$. Thus we obtain $(\partial q_f^*/\partial a)a = q_f^*$. The solution of this differential equation is given by (43).

For a complete hedge it is necessary that $q_f = a/f$ (see Proposition 1). Insert this quantity into the first order condition (42). Rearranging gives the definition in (44). The inequality follows from Jensen's inequality for strictly convex functions.

If we differentiate the first order condition implicitly we obtain

$$\frac{dq_f^*}{df} = -\frac{E\left[c^{r-1}y^{-r}\right]}{rE\left[c^{r-1}y^{-(1+r)}(I^k - f)^2\right]} + \frac{E\left[c^{r-1}y^{-(1+r)}(I^k - f)\right] q_f^*}{E\left[c^{r-1}y^{-(1+r)}(I^k - f)^2\right]} \; . \tag{57}$$

The first term on the RHS is the substitution effect, which is definitely negative. To derive this effect, we must compensate individuals with the amount $da = q_f^* df$. This compensation guarantees that the expected utility remains constant after a variation in f. The second term is the negative of the income effect dq_f^*/da and is less than or equal to zero. We derive the sign of this effect as follows: we obtain for the numerator with the help of the first order condition

$$E\left[c^{r-1}y^{-r}(y^{-1}-a^{-1})(I^k-f)\right]q_f^* \ . \tag{58}$$

This term is zero for $q_f^* = 0$. If $q_f^* > 0$ it follows for all realizations $I_s^k - f > 0$ that $1/y_s < 1/a$ and thus $(1/y_s - 1/a)(I_s^k - f) < 0$; corresponding to this it follows for all realizations $I_s^k - f < 0$ that $1/y_s > 1/a$ and thus $(1/y_s - 1/a)(I_s^k - f) < 0$. Finally the product is zero for all realizations with $I_s^k - f = 0$. In all these cases (58) is less than or equal to zero. One can use the same argument to show that (58) is also less than or equal to zero for $q_f^* < 0$.

With $q_f^* = 0$ one obtains immediately from the first order condition $f^0(r)$. We have to show that

$$E\left[\frac{c^{r-1}}{E[c^{r-1}]}\,I^k\right] \gtreqless E\left[I^k\right] \quad \Leftrightarrow \quad r \gtreqless 1 \ . \tag{59}$$

This is obvious for $r = 1$. In order to show this for $r \neq 1$ we make use of the concept of first order stochastic dominance. Let $f(c_s)$ be the density for the realizations $c_s \in \mathcal{C}$. Define

$$g(c_s, r) \equiv \frac{c_s^{r-1}}{E[c^{r-1}]} f(c_s) \ . \tag{60}$$

This is also a density function, which sums over all realizations to one. For $r_h > r_l \geqslant 1$ the distribution under r_h dominates the distribution under r_l first order, if

$$\sum_{c_s \leqslant \hat{c}} H(c_s) g(c_s, r_l) \leqslant 0 \qquad \forall \ \hat{c} \in [c_{min}, c_{max}] \tag{61a}$$

with

$$H(c_s) \equiv c_s^{r_h - r_l} - E\left[c^{r_h - 1}\right]/E\left[c^{r_l - 1}\right] \ . \tag{61b}$$

We have $H'(c_s) > 0$. Furthermore we have

$$H(c_{min}) < 0 \quad \text{because of} \quad E\left[(c/c_{min})^{r_l - 1}\right] < E\left[(c/c_{min})^{r_l - 1}\right] \tag{62a}$$

and

$$H(c_{max}) > 0 \quad \text{because of} \quad E\left[(c/c_{max})^{r_l - 1}\right] > E\left[(c/c_{max})^{r_l - 1}\right] \ . \tag{62b}$$

Finally,

$$\sum_{c_s \leqslant \hat{c}} H(c_s) g(c_s, r_l) \leqslant \sum_{c_s \leqslant c_{max}} H(c_s) g(c_s, r_l) = 0 \tag{63}$$

with strict inequality for $\hat{c} < c_{max}$. Thus we have shown first order stochastic dominance. It follows from this that the expected value for I^k with respect to the density $g(c_s, r_h)$ is greater than the expected value with respect to the density $g(c_s, r_l)$. One uses the same steps for $r_l < r_h \leqslant 1$.

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